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I. INTRODUCTION.

1. RULED quartic surfaces have been studied systematically by Cayley, Cremona, and Salmon, and to some extent by Rohn, Chasles, Schwartz, Reye, Voss, Holgate, and others. Of the quartic scrolls, Cremona,* in his excellent synthetic treatment, enumerates twelve species, while Cayley,† in the most complete and masterly analytical treatment of scrolls, divides these scrolls into ten species and gives a comparison of his species with those of Cremona, stating that the latter's two remaining species, though properly considered as distinct from the others, may be regarded as sub-forms of his seventh and ninth species. As we shall not have occasion here to take account of these sub-forms, we shall follow Cayley's classification. The other ruled quartic surfaces are: the developable quartic, or *torse*, whose edge of regression is a twisted cubic, and the quartic cones, which, although developable, must be considered separately. The developable quartic has been quite thoroughly studied by Cayley, Salmon, and others, but very little seems to have been done with the quartic cones, which, as will appear later, present some very interesting features.

The consideration of curves in space has been the subject of a great many articles by the most eminent geometricians, but curves on ruled quartic surfaces in particular have received little attention. As early as 1861, Chasles‡ gave a method of describing curves of order $4m + n$, on

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* Sulle superficie gobbe di quarto grado, Mem. della R. Istoria di Bologna, Series II., T. VIII.

† Second and Third Memoirs on Skew Surfaces, Otherwise Scrolls. Coll. Math. Papers, Vol. V. and Vol. VI., respectively, and Phil. Trans., 1863 and 1869, respectively.

‡ Description des courbes à double courbure de tous les ordres sur les surfaces réglées du troisième et du quatrième ordre. C. R. LIII, 884–889.

ruled quartic surfaces, by means of a pencil of surfaces of order m and groups of n generators in involution. In 1897, Rohn * treated some of the properties of curves on the general quartic surface, considering also the surfaces that can be passed through these curves and presenting some theorems regarding the residual intersection.

In 1883, Professor Story discovered a method by means of which he was able to classify all curves lying on a quadric surface and to give a formula for the number of intersections of these curves, thus obtaining, by a synthetic process, the results already found by Cayley.† Professor Story applied his method to the cubic scrolls, classifying all curves lying on these surfaces and obtaining a formula for the number of intersections of any two of these curves, analogous to that found for curves on quadrics.

By an extension of the analytical method of Cayley, Dr. Ferry ‡ succeeded most admirably in treating analytically the "Cubic Scroll of the First Kind," verifying the results of Professor Story for this surface, and in a paper soon to be published, Dr. Ferry has also verified, in the same way, Professor Story's results for the other cubic scroll.

It is the purpose of the present paper to consider the classification of curves on all ruled quartic surfaces; to find the formula for the number of intersections of any two curves that lie on the same ruled quartic surface; and to point out some of the most notable results obtained in the course of the investigation. The equations of many of the ruled quartic surfaces are so complicated that very serious difficulties arise when we attempt to treat them analytically, and it has been found most convenient to employ the synthetic method of Professor Story.

2. For convenience, we shall use the symbol $S^{(\nu)}$ to denote a surface of order ν, and $\Sigma^{(\mu)}$ to denote a ruled surface of order μ. $C^{(a)}$ will be used to denote a curve of order a lying on the ruled surface in question. By an arbitrary generator we shall mean any simple generator that bears no special relation to the curve in question, e. g. in considering a plane curve, it is any simple generator not lying in the plane of the curve. It must be proved, first of all, that every curve $C^{(a)}$ meets each generator of the surface on which it lies in a constant number of points, say

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* Die Raumcurven auf den Flächen IV[ter] Ordnung. Verhandlungen der K. Sächs. Gesell. der Wiss. zu Leipzig, 1897.

† On the curves situate on a surface of the second order. Coll. Math. Papers, Vol. V., and Phil. Mag., 1861, pp. 35–38.

‡ Geometry on the Cubic Scroll of the First Kind. Archiv for Mathematik og Naturvidenskab, B. XXI. Nr. 3, 1899.

a points, equal to the number in which it meets an arbitrary generator. (If the curve goes through a point common to all the generators, e. g. the vertex of a cone, this point is not counted as one of the a points on any generator.) This furnishes us a method for classifying all curves on $\Sigma^{(\mu)}$, and we shall use the symbol a_α to denote a curve of order a that meets each generator of the surface on which it lies in α points, α being a constant. Similarly, b_β will denote a curve of order b that meets each generator β times. Each generator is itself a plane curve of order 1, and since it is not met by any other generator except those lying in its plane, it is represented by 1_0. In case $\Sigma^{(\mu)}$ has more than one system of rulings, the lines of one system are chosen as the generators, with reference to which the classification is made for all curves on that surface, and the lines of any other system are regarded simply as curves of order 1.

We shall use the symbol (a_α, b_β) to represent the number of intersections of any two curves a_α and b_β on the ruled surface.

Professor Story * proved that for all curves lying on a quadric surface $(a_\alpha, b_\beta) = a\,\beta + b\,\alpha - 2\,\alpha\,\beta$, and for all curves lying on the cubic scrolls $(a_\alpha, b_\beta) = a\,\beta + b\,\alpha - 3\,\alpha\,\beta$, and stated that it is probably true that $(a_\alpha, b_\beta) = a\,\beta + b\,\alpha - \mu\,\alpha\,\beta$, for curves on a scroll of any order μ. It will be proved here that the formula is true for all ruled quartic surfaces, i. e. that we have

$$(1) \qquad (a\,, b_\beta) = a\,\beta + b\,\alpha - 4\,\alpha\,\beta,$$

and when we say that this formula holds for a certain curve we shall mean that it gives the number of intersections of this curve with any other curve on the scroll. It must be borne in mind that these formulæ for the *cones* of different orders give, in each case, the number of intersections of the curves aside from those at the vertex, since the vertex is not one of the α points on any edge. This will be proved for the quartic cones.

3. We shall first consider three general theorems,† which must be proved before the formula can be established. The first may be called the *fundamental theorem* and may be stated thus:

Theorem I. — If α be the number of points of $C^{(a)}$ on an arbitrary generator, there are α points of $C^{(a)}$ on each generator.

* On the Number of Intersections of Curves Traced on a Scroll of any Order. Johns Hopkins University Circulars, August, 1883.

† These theorems were given by Dr. Story in his lectures, October to December, 1899.

No general proof of this theorem has yet been found, and it must be proved for each of the ruled surfaces, separately.

From Theorem I are readily deduced the other two theorems, as follows:

Theorem II. — If a'_a is the complete intersection of $\Sigma^{(\mu)}$ and $S^{(\nu)}$, and if b_β is any curve on $\Sigma^{(\mu)}$ that has no component in common with a_α, then $(a_\alpha, b_\beta) = a\,\beta + b\,\alpha - \mu\,\alpha\,\beta$.

Proof. — The intersections of a_α and b_β are simply the intersections of $S^{(\nu)}$ and b_β and are in number equal to $b\,\nu$, i. e. $(a_\alpha, b_\beta) = b\,\nu$. Now since each generator meets $S^{(\nu)}$ in ν points, $\alpha = \nu$, also $a = \mu\,\nu = \mu\,\alpha$, and we have

$$a\,\beta + b\,\alpha - \mu\,\alpha\,\beta = \mu\,\alpha\,\beta + b\,\nu - \mu\,\alpha\,\beta = b\,\nu;$$

therefore
$$(a_\alpha, b_\beta) = a\,\beta + b\,\alpha - \mu\,\alpha\,\beta.$$

Theorem III. — If a_α is irreducible and the partial intersection of $\Sigma^{(\mu)}$ and $S^{(\nu)}$, $a'_{\alpha'}$ being the residual intersection, and if the formula holds for each irreducible component of $a'_{\alpha'}$ with an arbitrary curve b_β on $\Sigma^{(\mu)}$, it also holds for a_α with b_β.

Proof. — The residual $a'_{\alpha'}$ may break up into several curves, but b_β, being arbitrary, does not in general contain any part of the intersection of $\Sigma^{(\mu)}$ and $S^{(\nu)}$. If $a'_{\alpha'}$ is reducible, the order a' is the sum of the orders of the component curves, and the number of points α' in which any generator meets $a'_{\alpha'}$ is the sum of the numbers of points in which this generator meets the component curves. Since the complete intersection of $\Sigma^{(\mu)}$ and $S^{(\nu)}$ is $a_\alpha + a'_{\alpha'}$ we have, by Theorem II,

$$(a_\alpha + a'_\alpha\,,\, b_\beta) = (a + a')\,\beta + b\,(\alpha + \alpha') - \mu\,(\alpha + \alpha')\,\beta.$$

By supposition

$$(a'_\alpha, b_\beta) = a'\,\beta + b\,\alpha' - \mu\,\alpha'\,\beta.$$

Now the number of points in which b_β meets the complete intersection less the number in which it meets $a'_{\alpha'}$ must be the number of points in which it meets a_α; therefore

$$(a_\alpha, b_\beta) = a\,\beta + b\,\alpha - \mu\,\alpha\,\beta.$$

Corollary. — If the complete intersection of $\Sigma^{(\mu)}$ and $S^{(\nu)}$ consists of two curves and the formula holds for one of these curves it holds for the other also.

4. In order then to prove the formula for any $\Sigma^{(\nu)}$ it suffices first to prove Theorem I, and then to show that every curve on $\Sigma^{(\mu)}$ can be cut out by an $S^{(\nu)}$ such that the residual is a curve, or is composed of curves,

for which the formula holds. Now the formula holds for every generator, i. e. for a 1_0, since 1_0 meets b_β in β points, and the formula gives $(1_0, b_\beta) = 1.\beta + b.0 - \mu.0.\beta = \beta$. Therefore, if every conic can be cut out by an $S^{(\nu)}$ such that the residual is nothing but generators, if every cubic curve can be cut out by an $S^{(\nu)}$ such that the residual consists entirely of conics or generators or both, and in general, if every a_a on $\Sigma^{(\mu)}$ can be cut out by an $S^{(\nu)}$ such that the residual * is of order less than a or is composed of curves of orders less than a, the formula is true.

5. For certain species of $C^{(a)}$'s it may be possible to choose ν smaller than for certain other species, e. g. all the quartic curves lying on a quadric surface can certainly be cut out by cubic surfaces, but the "quartics of the first kind" can also be cut out by quadric surfaces.

We shall first determine the lowest value of ν for which we can be certain that an $S^{(\nu)}$ will cut out any species of $C^{(a)}$. This can be done for a surface of any order, μ, without difficulty, but since we are here going to treat the ruled quartics only, we shall consider the case of $\mu = 4$ only.

$S^{(\nu)}$ is determined by $\frac{1}{6}(\nu + 1)(\nu + 2)(\nu + 3) - 1$ arbitrary points.† When $\nu \geqq 5$ we must take care that $S^{(\nu)}$ does not break up into $\Sigma^{(4)}$ and a surface of order $\nu - 4$, i. e. of the points necessary to determine $S^{(\nu)}$ we must take one more than enough to determine a surface of order $\nu - 4$ as not lying on $\Sigma^{(4)}$; also, we must take $a\,\nu + 1$ points of $S^{(\nu)}$ on $C^{(a)}$ in order that $S^{(\nu)}$ may contain this curve; so that for $\nu \geqq 5$ the number of arbitrary points of $\Sigma^{(4)}$ through which we can make $S^{(\nu)}$ pass is

$$(2) \quad \ldots \quad \tfrac{1}{6}(\nu+1)(\nu+2)(\nu+3) - 1 - \tfrac{1}{6}(\nu-3)(\nu-2(\nu-1) - (a\,\nu+1) \\ = 2\,\nu^2 - a\,\nu.$$

For $\nu = 4$ we must take one point of $S^{(\nu)}$ not lying on $\Sigma^{(4)}$, but then the term $\frac{1}{6}(\nu - 3)(\nu - 2)(\nu - 1) = 1$; for $\nu = 1, 2,$ or 3 we do not have to take any points of $S^{(\nu)}$ off $\Sigma^{(4)}$, but then the term $\frac{1}{6}(\nu - 3)(\nu - 2)(\nu - 1) = 0$; therefore formula (2) gives the number of arbitrary points for all values of ν, when $\mu = 4$. We have, therefore, $2\,\nu^2 - a\,\nu \geqq 0$, which gives at once $\nu \geqq \frac{a}{2}$; so that, for the lowest value of ν, we have $\nu = \frac{a}{2}$ when a is even, and $\nu = \frac{a+1}{2}$ when a is odd. In some cases it has been found more convenient, and apparently necessary, to take ν

* The letter r, when used, shall always denote the order of the total residual.

† Salmon's Geom. of Three Dimensions, Chap. XI.

greater by one than this lowest value, in order to be able to make the residual consist of curves of orders less than a.

In determining the number of points at our disposal, given by formula (2), we have said nothing about multiple points on $C^{(a)}$, but have supposed that the points of $S^{(\nu)}$ that had to be taken on $C^{(a)}$ were ordinary points on this curve, and we shall always consider ν chosen without regard to multiple points on $C^{(a)}$. If a surface be made to pass through an ordinary point of a curve it meets the curve once at that point, and therefore we have to make $S^{(\nu)}$ pass through $a\,\nu + 1$ ordinary points of $C^{(a)}$ in order to make $S^{(\nu)}$ contain $C^{(a)}$; but if $C^{(a)}$ has a double point, any surface through this double point will meet the curve twice there, and therefore, if we make $S^{(\nu)}$ pass through this double point (which counts for only one point in the determination of $S^{(\nu)}$), we have to make it pass through only $a\,\nu - 1$ other ordinary points of $C^{(a)}$ in order to make it contain $C^{(a)}$, since $C^{(a)}$ will then intersect $S^{(\nu)}$ in $a\,\nu - 1 + 2 = a\,\nu + 1$ points. Consequently, when $C^{(a)}$ has a double point, only $a\,\nu$ of the points necessary to determine $S^{(\nu)}$ need be taken on $C^{(a)}$ if we take the double point to be one of these: this is one less than the number of points of $S^{(\nu)}$ taken on $C^{(a)}$ in deducing formula (2), above; and therefore, when $C^{(a)}$ has a double point, we shall have at our disposal one point more than the number given by formula (2). In like manner, if $C^{(a)}$ has an m-tuple point, a surface $S^{(\nu)}$ through that point meets $C^{(a)}$ m times there, and we need only make $S^{(\nu)}$ pass through $a\,\nu - (m-1)$ other ordinary points in order that it shall contain $C^{(a)}$; and, consequently, when $C^{(a)}$ has an m-tuple point, we shall have at our disposal $m - 1$ points more than the number given by formula (2).

In accordance with this principle, it is evident that, if $\nu + 1$ branches of $C^{(a)}$ meet any line L* (i. e. if $\nu + 1$ of the points of intersection of

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* It is necessary to observe here a very important fact, which is often overlooked, viz., if a curve has an m-tuple point P and the m tangents to the curve at P all lie in the same plane, a surface on which the curve lies may have this point P as an ordinary point, and any line L through P, that does not lie in the tangent plane, meets this surface only once at P; but there are m branches of the curve that meet L at P and a plane through L meets the curve m times at P. In general, if the m tangents at P do not all lie in the same plane, P will be a multiple point on any surface that contains the curve, the multiplicity k of P on any such surface being at least equal to the order of the cone of lowest order that can be passed through the m tangents, and this lowest order is always less than m. Then any line L that does not lie on the tangent cone to the surface at P meets the surface only k times at P, while any edge of the cone meets the surface $k + 1$ times there.

$C^{(a)}$ with an arbitrary plane through the line L lie on L), we can then make L lie on the surface $S^{(\nu)}$ that cuts out $C^{(a)}$ and still have left at our disposal the number of points given by formula (2). For, if the $\nu+1$ points of $C^{(a)}$ on L are all ordinary points, L meets $S^{(\nu)}$ $\nu+1$ times and therefore lies on $S^{(\nu)}$; but, if $C^{(a)}$ has an m-tuple point and $\nu-m+1$ other points on L, even if L meets $S^{(\nu)}$ only once at this multiple point, $m-1$ points not included in formula (2) are still at our disposal and may be taken on L, so that $\nu+1$ points of $S^{(\nu)}$ lie on L and L will therefore lie on $S^{(\nu)}$. Therefore, in considering whether a line can be made to lie on $S^{(\nu)}$ or not, we need not take account of the multiple points of $C^{(a)}$ that lie on this line, but may regard the line as meeting $S^{(\nu)}$ in points of $C^{(a)}$ equal in number to the number of points on the line in which an arbitrary plane through the line meets $C^{(a)}$.

6. We have seen that, when certain theorems can be proved, formula (1), p. 21, gives the number of intersections of any two curves on the same ruled surface. In special cases, where the curves bear a particular relation to one another, and in most cases where the multiple curve is involved, the result given by this formula requires a special interpretation, namely: if two curves on $\Sigma^{(4)}$ pass through the same point of the multiple curve, any branch of either curve is regarded as intersecting only those branches of the other curve that lie on the same sheet with it, and two branches that pass through the same point of the multiple curve are not regarded as intersecting at that point if they lie on different sheets of $\Sigma^{(4)}$ there. In particular, two generators through the same point of the multiple curve are not regarded as intersecting, when considered as loci on that quartic surface.

The double curve on a ruled quartic cannot be of order greater than three, and therefore a plane section can never have more than three double points on the double curve. If the plane section has a double point not on the double curve, this double point is a point of tangency of the plane, and, since a tangent plane to a ruled surface contains the generator through the point of tangency, the section must be a degenerate quartic curve having at least one generator as a component. Therefore the section of a ruled quartic by a plane can never consist of two proper conics; for the section would then have four double points, one of which must be a point of tangency of the plane, and therefore the plane would cut out a generator.

Cayley * uses the general symbol $S(m, n, p)$ to denote a scroll gener-

* Second Memoir on Skew Surfaces, Otherwise Scrolls. Coll. Math. Papers, Vol. V., and Phil. Trans., 1863

ated by a line that meets each of three curves of orders m, n, and p once, $S(m^2, n)$ to denote a scroll generated by a line that meets a curve of order m twice and a curve of order n once, and $S(m^3)$ to denote a scroll generated by a line that meets a curve of order m three times. In his symbols for the quartic scrolls he has also used a subscript, in most cases, to denote the order of multiplicity of the curve on the scroll; but he has not, in all cases, adhered to his general method, and it seems best, while preserving his classification, to change his symbols, making them conform to his general rule for such symbols.

II. Quartic Scroll, with a Triple Linear Director and a Simple Linear Director, $S(1_3, 1, 4)$. (Cayley's Third Species, $S(1_3, 1, 4)$.)

1. This scroll has three sheets through the triple linear director, which we shall denote by T, and T is scrolar * on each sheet.

Through each point of T pass three generators, one on each sheet, and if we pass a plane through two of these generators it will also contain the simple director, since each generator meets the simple director once, and therefore the third generator at the point lies in this same plane, for it meets it once at the point and once on the simple director; i. e. any plane through the simple director meets the scroll in this director and in three generators that intersect in the point where the plane meets T.

2. *Proof of Theorem I.* — Pass a plane through T; it meets the scroll in T and one generator and meets $C^{(a)}$ in a points. Now if we revolve the plane about T it will cut out, in succession, each generator of the scroll, and since the plane always meets $C^{(a)}$ in the same number of points, say τ points, on T, it meets $C^{(a)}$ in the same number of points, say α points on each generator, where $\tau + \alpha = a$. Since three generators lie in a plane $\alpha \overline{\overline{<}} \dfrac{a}{3}$ and $\tau \overline{\overline{>}} \dfrac{2a}{3}$.

3. *Plane Curves.* — A plane that does not pass through any line on the scroll, i. e. an arbitrary plane, meets the scroll in a plane quartic curve having a triple point on T, and since an arbitrary generator meets the plane once, every plane quartic is a 4_1.

A plane through one and only one generator cuts out a plane cubic having a double point on T, through which the generator passes, making

* Cayley calls a line *scrolar* on a surface when the tangent plane to the surface is different at each point along the line.

a triple point on the complete intersection; the generator meets the cubic again where the plane is tangent to the scroll. Since an arbitrary generator meets the plane once and does not meet the generator lying in that plane, it meets the plane cubic once, and therefore every plane cubic is a 3_1. If a plane cuts out a proper conic, it must also cut out another conic, which must be an improper conic consisting of two lines through a point of the proper conic, since the section by the plane must have a triple point on T; but the only lines on the scroll that pass through a point of T, besides T itself, are generators, and we have seen that a plane through two generators also cuts out a third generator and the simple director; therefore, there are no conics on this scroll. The triple director T is met once by each generator, and is, therefore, a 3_1. The simple director is met once by each generator, and is, therefore, a 1_1.

Each of these plane curves is either the complete intersection of the scroll and a plane, or else the residual intersection consists entirely of generators, and therefore by Theorems II and III, formula (1) holds for every plane curve on the scroll.

A plane quartic has a branch on each sheet where it crosses T, and therefore meets T three times, as the formula shows,

$$(4_1, 3_1) = 4 + 3 - 4 = 3.$$

A plane cubic has a branch on each of two sheets where it crosses T, and therefore meets T twice, $(3_1, 3_1) = 3 + 3 - 4 = 2$. The simple director does not meet T, $(1_1, 3_1) = 1 + 3 - 4 = 0$. The simple director meets a plane once, and therefore meets a plane quartic once, $(1_1, 4_1) = 1$, but since it meets each generator once, it cannot meet a plane cubic, $(1_1, 3_1) = 0$. Two plane cubics intersect in two points, on the line of intersection of their planes, $(3_1, 3_1) = 3 + 3 - 4 = 2$, the other two points, where this line meets the scroll, being the two points where each cubic is met by the generator that lies in the plane of the other. A plane cubic and a plane quartic intersect in three points on the line common to their planes, $(3_1, 4_1) = 3 + 4 - 4 = 3$, the fourth point where this line meets the scroll being the point where the generator in the plane of the cubic meets the plane quartic. Two plane quartics meet in four points on the line of intersection of their planes, $(4_1, 4_1) = 4 + 4 - 4 = 4$.

4. *Twisted Cubic* 3_1. — We saw that when a is odd we can take $\nu = \dfrac{a+1}{2}$; so, for the twisted cubic, $\nu = 2$, and by formula (2), p. 23, we have two points at our disposal in the determination of this quadric,

$S^{(2)}$, that cuts out the cubic. Since $\alpha \overline{\overline{<}} \frac{a}{3}$ we have $\alpha = 1$, and every twisted cubic is a 3_1; therefore $\tau = 3 - 1 = 2$, i. e. the twisted cubic meets T in two points, which must be distinct, since a twisted cubic cannot have a double point. Therefore the quadric that cuts out the twisted cubic meets T in two points on this curve, and since we can make the quadric pass through any two points we please that are not on the curve, we can make it pass through another point of T, and it will then contain T. The residual intersection, which is of order 5, then consists of T, which counts for three lines, and two generators, since there are no conics on the scroll, and, moreover, each generator meets the quadric once on T and once on the twisted cubic, and cannot meet it again without lying on it; and if a conic or the simple director formed part of the residual, an infinite number of generators would lie on the quadric, which is impossible. Since formula (1) holds for T and the generators, by Theorem III it holds for every twisted cubic.

A plane through the simple director cuts out three generators and meets the twisted cubic three times, once on each generator, and therefore the twisted cubic does not meet the simple director.

5. *Twisted Quartic*, 4_1. — We have $a = 4$, $\nu = \frac{a}{2} = 2$, and $r = 4$, where r is the order of the residual; $\alpha = 1$, and every twisted quartic is a 4_1. Hence $\tau = 3$, and if the quartic has no double point on T there must be three distinct points of the curve on T, i. e. T meets the quadric that cuts out the quartic three times, and therefore lies on it. If the quartic has a double point on T, it is a "quartic of the first kind," and we can pass a quadric through it and through any arbitrary point not on the curve; the quadric already meets T twice on the quartic curve, and if we make it pass through another point of T, T will lie entirely on it. In any case, the twisted quartic can be cut out by a quadric such that the residual will consist of T and one generator, and, since formula (1) holds for T and all generators, it holds for every twisted quartic; if T lies on the quadric the simple director cannot form part of the residual, since each generator already meets the quadric once on T and once on the twisted quartic. The twisted quartic has a point on each generator, and therefore meets the simple director once, $(4_1, 1_1) \doteq 1$.

6. *Twisted Curves, in general.* — When a is odd we take $\nu = \frac{a+1}{2}$, whence $r = a + 2$; and when a is even we take $\nu = \frac{a}{2}$, whence $r = a$. We saw that $\alpha \overline{\overline{<}} \frac{a}{3}$ and $\tau \overline{\overline{>}} \frac{2a}{3}$ where τ is the number of points on T,

in which a_a is met by a plane through T. For $a = 3$, we saw that T could be made to lie on $S^{(2)}$, the quadric that cuts out the twisted cubic. Now for $a > 3$, $\nu = \frac{a+1}{2} < \tau$ when a is odd and $\nu = \frac{a}{2} < \tau$ when a is even. It follows at once, therefore, from what was said on p. 24, that we can always make T lie on $S^{(\nu)}$. Therefore when a is odd the residual can be made to consist of T and a curve of order $r - 3 = a - 1$, and when a is even the residual can be made to consist of T and a curve of order $r - 3 = a - 3$. Therefore, by Theorem III, if formula (1) holds for every curve of order less than a, it holds for every curve of order a; but we have proved that it holds for all plane curves and for all twisted curves of orders 3 and 4; it therefore holds for every curve of order 5 and therefore for every curve of order 6 and so on, and it therefore holds for every curve on the scroll.

7. The above proof is also applicable to the *Quartic Scroll, with a two-fold* 3 (+1) *-tuple linear director,* $S\,(\overline{1_3, 1}, 4)$. (*Cayley's Sixth Species.*) This scroll is, in fact, the limiting case of the scroll just considered, where the simple director has moved up into coincidence with the triple director. Cayley denotes this symbolically by drawing a bar over the two 1's.

The triple linear director on this scroll is torsal along one of the three sheets through it, i. e. the tangent plane to this sheet, along this director, is the same for every point of this director; the generator, lying on this sheet, that is cut out by this tangent plane, coincides with the simple linear director, and is regarded as intersecting the triple linear director at the point where the other two generators cut out by this tangent plane, one on each sheet, intersect.

III. Quartic Scroll, with a Triple Linear Director, $S\,(1_3, 2, 2)$. (Cayley's Ninth Species, $S\,(1_3)$.)

1. The triple director, T, is scrolar on each of the three sheets that pass through it, and this scroll differs from the Quartic Scroll $S\,(1_3, 1, 4)$ in not having a simple linear director, in consequence of which we have, on this scroll, three generators through each point of T that do not lie in the same plane. The plane of any two of these three generators meets the scroll otherwise in a conic that passes through their point of intersection on T, making up the triple point of the complete intersection. Therefore, there are three conics through each point of T, one in each of the three planes that contain two generators through the point.

The proof of *Theorem I* is the same as for the Quartic Scroll $S(1_3, 1, 4)$, p. 26. Since two generators lie in a plane, $\alpha \overline{\overline{<}} \frac{a}{2}$.

2. *Plane Curves.* — As before, each plane curve is met once by an arbitrary generator, i. e. $\alpha = 1$ for any plane curve. There is no 1_1 on this scroll. Each conic is a 2_1, the triple director T is a 3_1, and the other plane curves, 3_1 and 4_1, are the same as for the Quartic Scroll $S(1_3, 1, 4)$. Every plane curve is either the complete intersection of the scroll by its plane or else the residual is composed of generators, and therefore, by Theorems II and III, formula (1) holds for every plane curve.

Two conics do not intersect; even if they pass through the same point they lie on different sheets, and cannot be regarded as intersecting on the scroll; the formula gives $(2_1, 2_1) = 2 + 2 - 4 = 0$; the line of intersection of the planes of the two conics meets the scroll in the four points where the two generators in the plane of either conic meet the other conic. In the plane of a conic each of the two generators that lie in that plane meets the conic on T and at one other point where the plane is tangent to the scroll; therefore the plane of every conic is a double tangent plane to the scroll. T meets each conic once, $(3_1, 2_1) = 3 + 2 - 4 = 1$. A conic meets a plane cubic once, $(2_1, 3_1) = 1$, and meets a plane quartic twice, $(2_1, 4_1) = 2 + 4 - 4 = 2$.

3. *Twisted Curves.* — Since $\alpha \overline{\overline{<}} \frac{a}{2}$, we have for the twisted cubic $\alpha = 1$ and $\tau = 2$, where τ is the number of points of intersection of the curve and an arbitrary plane through T that lie on T. By the same reasoning as that employed on page 28, we see that T can be made to lie on the quadric that cuts out the twisted cubic, and that formula (1) holds for every twisted cubic.

When a is odd we take $\nu = \frac{a+1}{2}$. Then, since two generators lie in a plane and α is an integer, $\alpha \overline{\overline{<}} \frac{a-1}{2}$ and $\tau \overline{\overline{>}} \frac{a+1}{2}$; but by formula (2) we have $\frac{a+1}{2} = \nu$ points at our disposal in the determination of $S^{(\nu)}$ and therefore we can make T lie on $S^{(\nu)}$; the residual will then consist of T and a curve of order $r - 3 = a + 2 - 3 = a - 1$.

When a is even we take $\nu = \frac{a}{2}$; $\alpha \overline{\overline{<}} \frac{a}{2}$ and $\tau \overline{\overline{>}} \frac{a}{2}$. If $\alpha < \frac{a}{2}$, $\tau > \frac{a}{2}$, i. e. $\tau > \nu$, and it follows from what was said on page 24 that T can be made to lie on $S^{(\nu)}$; the residual will then consist of T and a curve of order $a - 3$. If $\alpha = \frac{a}{2} = \nu$, every generator meets $S^{(\nu)}$ in ν points, which

are points of a_α; if, then, any generator meets $S^{(\nu)}$ in an additional point it must lie on $S^{(\nu)}$, and therefore if any generator has on it a point of the residual it lies on $S^{(\nu)}$ and is itself a part of the residual; therefore, when $\alpha = \frac{a}{2}$ the residual consists entirely of generators which are a in number, since the residual is of order a; and if the curve has no multiple points on T, there are $\frac{a}{2}$ pairs of generators that pass through the $\frac{a}{2}$ points where a_α meets T.

We have shown then that every twisted curve of order a can be cut out by an $S^{(\nu)}$ such that the residual will consist of curves of orders less than a, and it therefore follows, as on page 29, that formula (1) holds for every curve on the scroll.

IV. Quartic Scroll, with two Double Linear Directors and with a Double Generator, $S(1_2, 1_2, 2)$. (Cayley's Second Species, $S'(1_2, 1_2, 4)$.)

1. Let us call the double linear directors D and D'; they do not intersect, and a plane through either of them cuts out also two generators that intersect in the point where the plane meets the other director, i. e. any generator A meets a definite generator B on D and another definite generator E on D', so that A and B lie in a plane through D' and A and E lie in a plane through D, while B and E do not meet. In a special form of this scroll four generators may form a gauch-quadrilateral having two vertices on each double director, e. g. taking the generators above, if A and B meet D at the point P, A and E meet D' at the point R, B meets D' at the point S, and E meets D at the point Q, the scroll may be of such a form that a generator F will pass through Q and S, as can easily be shown analytically. It is also very probable that there are special forms of this scroll on which any even number of generators, greater than four, form a gauch-polygon, but it is not the purpose of this paper to discuss these special forms. The double generator, which we will denote by G, arises from the fact that the plane quartic directing curve has three double points, one on each of the double directors and one through which G passes; a plane through G and either double director does not meet the scroll again.

2. *Proof of Theorem I.* — On any quartic scroll where the double curve is a twisted cubic, either proper or degenerate, we can prove Theorem I by passing a quadric through this twisted cubic. In the

present case the twisted cubic is degenerate, consisting of D, D', and G. Let us pass a quadric through eight points, three on D, three on D', one on G, and one on any generator A, the last two points not being on D or D'; then D, D', G, and A will all lie on the quadric and count for 7 lines in the intersection of the scroll and quadric, and therefore the quadric cuts out one more generator; the quadric passes through eight fixed points and we can make it pass through an arbitrary ninth point, so if we vary this ninth point continuously the quadric will cut out, in succession, each generator of the scroll. Now $C^{(a)}$ meets the quadric in $2a$ points, of which a fixed number lie on D, D', G, and A, and therefore the same number of points of $C^{(a)}$, say α points, lie on each generator. It is evident that there must be α points on A, for if any other generator be chosen, through which the quadric is always to pass, then there is the same number of points, α, on A, as on each of the other generators. Since we can pass a plane through D and two generators, there are $a - 2\alpha$ points of a_α on D, and, similarly, there are $a - 2\alpha$ points of a_α on D'. A plane through D and G meets the scroll in these two lines only, and there are, therefore, 2α points of a_α on G, as is otherwise evident from the fact that G counts for two generators. Since a twisted curve of order a cannot have a points on any line, $2\alpha < a$ or $\alpha < \frac{a}{2}$ for every twisted curve on the scroll.

3. *Plane Curves.* — Each double director is met once by any generator and is therefore a 2_1. No generator can meet G; for, suppose a generator A does meet it; then A meets either D or D' in a point different from that in which G meets it, and therefore the plane through G and A contains also D or D', making the order of the complete intersection of the plane and scroll as great as 5, which is impossible. G is therefore a 2_0. Then any plane through G, that does not contain D or D', meets the scroll in a proper conic that does not meet either double director, since the section has only a double point on each double director; since each generator meets the plane once and does not meet G, each conic is a 2_1, and the conics and D and D' are the only curves on the scroll for which $\alpha = \frac{a}{2}$. A plane through one and only one generator cuts out a plane cubic, a 3_1, having a double point on G and passing once through the two points where the generator in the plane meets D and D'; the generator meets the cubic again where the plane is tangent to the scroll. A plane that does not contain a line of the scroll cuts out a plane quartic, a 4_1, having three double points, one on G and one on each

double director. Every plane quartic is the complete intersection of its plane and the scroll, and therefore formula (1) holds for it (Theorem II). A plane cubic is cut out by a plane through a single generator, and D and D' are cut out by planes through two generators, and therefore, by Theorem III, formula (1) holds for every plane cubic and for D and D'. We can cut out G by a plane through D, and, since formula (1) holds for D, by corollary to Theorem III, it holds for G. Each conic is cut out by a plane through G, and, since formula (1) holds for G, it holds for each conic. Therefore formula (1) holds for every plane curve on the scroll.

Either double director and G lie on both sheets of the scroll through them, respectively, and G, therefore, intersects either double director twice, once on each sheet, $(2_0, 2_1) = 2$. A plane quartic has a branch on each sheet, at each of the three points where it meets D, D', and G, and it therefore intersects each of these lines twice, once on each sheet, as the formula shows, $(4_1, 2_1) = 2$ and $(4_1, 2_0) = 2$. A plane cubic meets G twice, since it has a branch on each sheet where it crosses G, $(3_1, 2_0) = 2$, but it meets each double director once, since it has a branch on one sheet only, where it crosses either double director, $(3_1, 2_1) = 3 + 2 - 4 = 1$. The plane of a conic passes through G and is tangent to the scroll at two points along G, one on each sheet; these points of tangency are the two points of intersection of the conic and G, $(2_1, 2_0) = 2$, and the conic has a branch on each sheet; one point of tangency lies on the finite segment of G, between D and D', and the other lies on the infinite segment, so that, as we turn the plane about G in one direction, these two points both move toward the intersection of G and D and coincide at this intersection, forming a *pinch point*, when the conic becomes D, i. e. a line on each sheet; if we turn the plane in the other direction, or continue to turn it in the same direction after it cuts out D, the two points of tangency will both move toward D' and will coincide at the pinch point, the intersection of G and D', when the conic becomes D'.

It is easy to see, by the aid of formula (1), how the other plane curves intersect.

4. *Twisted Cubic*, 3_1. — We have seen that $\alpha < \frac{a}{2}$ for all twisted curves, and, consequently, every twisted cubic is a 3_1. Also, if δ be the number of points of the curve on D or D', $\delta = a - 2\alpha = 1$ for the cubic. The twisted cubic is cut out by a quadric $\left(\nu = \frac{a+1}{2}\right)$, and we can make the quadric contain D, since, by formula (2), we have two

points at our disposal in determining the quadric. Every generator then meets the quadric once on the twisted cubic and once on D, and cannot meet it again without lying on it; therefore, the residual consists of D and three generators. Since formula (1) holds for D and the generators, it holds for every twisted cubic on the scroll (Theorem III).

5. *Twisted Quartic* 4_1. — Since $\alpha < \frac{a}{2}$, every twisted quartic is a 4_1; $\delta = a - 2\alpha = 2$; i. e. a plane through D or D' meets the twisted quartic twice on that line. If the curve is a "quartic of the first kind" it may have a double point on D, D', or G, but in any case it has two distinct or consecutive points on one of the double directors, say D, and since we can pass a quadric through a "quartic of the first kind" and any arbitrary point, we may take this arbitrary point on D, and D will then lie on the quadric; if the quartic has no double point on G the residual will then consist of D and G, and if the quartic has a double point on G the residual will consist of D and two generators, since each generator will then meet the quadric once on D and once on the quartic. If the curve is a "quartic of the second kind," it is more convenient to cut it out by a cubic surface, i. e. we take $\nu = 3$; the residual then is of order 8, and by formula (2) we have six points at our disposal in the determination of $S^{(3)}$; since this quartic has no double point, it meets both D and D' in two distinct or consecutive points, and if we put two more points of $S^{(3)}$ on each of these double directors they will both lie on $S^{(3)}$; then G will meet $S^{(3)}$ once on D, once on D', and twice on the quartic, and will, therefore lie on $S^{(3)}$; each generator meets S once on D, once on D', and once on the quartic, and, since we still have two points at our disposal, we can make two generators lie on $S^{(3)}$; the residual will then consist of D, D', G, and two generators. Since formula (1) holds for D, D', G, and the generators, it holds for every twisted quartic on the scroll (Theorem III).

6. *Twisted Curves in General.* — When a is odd we take $\nu = \frac{a+1}{2}$, and the order of the residual is $r = a + 2$. We have seen that $\alpha < \frac{a}{2}$ or $1 \leqq \alpha \leqq \frac{a-1}{2}$, and that there is the same number of points, say δ points, on each double director, where $\delta = a - 2\alpha$, i. e. $1 \leqq \delta \leqq a - 2$. It has also been shown, p. 25, that we need not consider whether a_α has multiple points on D and D' or not. By formula (2), we have $\frac{a+1}{2}$ points at our disposal, in the determination of $S^{(\nu)}$, and therefore we can

make both D and D' lie on $S^{(\nu)}$ if $\delta + \frac{a+1}{4} \overline{\gtrless} \frac{a+1}{2} + 1$, i.e. if $\delta \overline{\gtrless} \frac{a+5}{8}$ or $\alpha \overline{\lessgtr} \frac{3a-5}{8}$. Therefore for $\alpha \overline{\lessgtr} \frac{3a-5}{8}$ the residual can be made to consist of D, D', and a curve of order $r - 4 = a - 2$. When $\alpha > \frac{3a-5}{8}$, at least one point of a_α lies on D, and since we have $\frac{a+1}{2}$ points at our disposal, we can make D lie on $S^{(\nu)}$, and the residual will then consist of D and a curve of order $r - 2 = a$, say a_ρ, where ρ is the number of points of this curve on each generator; now formula (1) holds for D, and if it holds for a_ρ it will hold for a_α (Theorem III), so we need only consider the curve a_ρ; each generator meets $S^{(\nu)}$ in $\nu = \frac{a+1}{2}$ points, once on D, α times on a_α, and ρ times on a_ρ, so that

$$\rho = \frac{a+1}{2} - 1 - \alpha < \frac{a+1}{2} - 1 - \frac{3a-5}{8} \text{ or } \rho < \frac{a+1}{8},$$

i.e. $\rho < \frac{3a-5}{8}$ for $a \overline{\gtrless} 3$, and a_ρ is therefore a curve like that considered above, that can be cut out by an $S^{(\nu)}$ such that the residual consists of curves of orders less than a.

When a is even, it is convenient to separate the curves into two divisions, according as $\frac{a}{2}$ is odd or even. If $\frac{a}{2}$ is odd we take $\nu = \frac{a}{2}$; then $r = a$, $\alpha < \frac{a}{2}$, and δ must be even since $\delta = a - 2\alpha$; if $\delta \overline{\gtrless} \frac{a}{2} + 1$, i.e. if $\alpha \overline{\lessgtr} \frac{a-2}{4}$, both D and D' lie on $S^{(\nu)}$ (or can be made to lie on $S^{(\nu)}$), and the residual consists of D, D', and a curve of order $r - 4 = a - 4$; if $\alpha > \frac{a-2}{4}$ the residual is a curve of order a, say an a_ρ, where $\rho = \frac{a}{2} - \alpha < \frac{a}{2} - \frac{a-2}{4}$, or $\rho < \frac{a+2}{4}$; but $\frac{a+2}{4}$ is an integer and ρ is an integer, and therefore $\rho \overline{\lessgtr} \frac{a+2}{4} - 1$, i.e. $\rho \overline{\lessgtr} \frac{a-2}{4}$; therefore, a_ρ is a curve like that just considered, that can be cut out by an $S^{(\nu)}$ such that the residual will consist of curves of orders less than a. If $\frac{a}{2}$ is even we first take $\nu = \frac{a}{2}$, for which $r = a$; δ is even, and if $\delta \overline{\gtrless} \frac{a}{2} + 2$, i.e. if $\alpha \overline{\lessgtr} \frac{a-4}{4}$, both D and D' lie on $S^{(\nu)}$ (or can be made to lie on

$S^{(\nu)}$), and the residual then consists of D, D', and a curve of order $a - 4$; if $\alpha > \frac{a-4}{4}$, $\alpha \gtreqless \frac{a}{4}$, and when $\alpha > \frac{a}{4}$ the residual is an a_ρ, where $\rho = \frac{a}{2} - \alpha < \frac{a}{2} - \frac{a}{4}$ or $\rho \lesseqgtr \frac{a-4}{4}$, and a_ρ is therefore a curve that can be cut out by an $S^{(\nu)}$ such that the residual will consist of curves of orders less than a; finally, if $\alpha = \frac{a}{4}$ we take $\nu = \frac{a}{2} + 1$; then $r = a + 4$ and $\delta = \frac{a}{2}$; if then we put two more points of $S^{(\nu)}$ on D and two more on D', both of these double directors will lie on $S^{(\nu)}$, and this we can always do, since, by formula (2), we have $a + 2$ points at our disposal in the determination of $S^{(\nu)}$ and $a \gtreqless 4$; then G will meet $S^{(\nu)}$ once on D, once on D', and $2\,\alpha = \frac{a}{2}$ times on a_α, and will therefore lie on $S^{(\nu)}$; each generator will meet $S^{(\nu)}$ once on D, once on D', and $\frac{a}{4}$ times on a_α, and, consequently, if we put $\frac{a}{4}$ more points of $S^{(\nu)}$ on any generator it will lie on $S^{(\nu)}$; now we still have at our disposal $a - 2 \gtreqless 2\left(\frac{a}{4}\right)$ points, since $a \gtreqless 4$, and therefore we can make two generators lie on $S^{(\nu)}$; therefore the residual can be made to consist of D, D', G, two generators, and a curve of order $r - 8 = a - 4$.

Therefore, on this scroll, we may divide all twisted curves into two groups, viz. group (1), those that may be cut out by an $S^{(\nu)}$ such that the residual consists of curves of orders less than a, and group (2), those that may be cut out by an $S^{(\nu)}$ such that the residual is a curve of group (1), with or without D. Now we have seen that formula (1) holds for all plane curves and for all twisted curves of order 3 or 4; it therefore holds for all curves of order 5 of group (1) (Theorem III), and it therefore holds for all curves of order 5 of group (2); it then holds for all curves of order 6 of group (1), and therefore for all curves of order 6 of group (2), and so on. Therefore formula (1) holds for every curve on the scroll.

7. The above proof is also applicable to the *Quartic Scroll, with a two-fold* 2 (+ 2)*-tuple linear director, and with a double generator* $S(\overline{1_2, 1_2}, 2)$, (*Cayley's Fifth Species,* $S'(\overline{1_2, 1_2}, 4)$), for this scroll is simply the limiting case of the scroll just considered, where one of the double directors has moved up into coincidence with the other. A plane quartic has a tac-node, where it meets the two-fold director and has a double point on the double generator. A plane cubic, regarded as lying in the

plane, is tangent to the generator in its plane where it meets the two-fold director, but regarded as lying on the scroll, it does not meet the generator there, for they lie on different sheets, and the generator meets the cubic in one point only, where the plane is tangent to the scroll — the formula giving $(1_0, 3_1) = 1$. The system of conics is the same as on the scroll just considered. The two-fold director may be regarded as a 4_2, since it has two lines on each of the two sheets through it and is met twice by each generator. A plane quartic has a branch on each sheet, and therefore meets the two-fold director four times, $(4_1, 4_2) = 8 + 4 - 8 = 4$, while a plane cubic has a branch on one sheet only, and therefore meets this director twice, $(3_1, 4_2) = 6 + 4 - 8 = 2$.

V. Quartic Scroll, with two Double Linear Directors and without a Double Generator, $S(1_2, 1_2, 4)$. (Cayley's First Species, $S(1_2, 1_2, 4)$.)

1. We shall call the double directors D and D'. They do not intersect, and if we pass a plane through either it cuts out two generators. The scroll is similar in many respects to the Quartic Scroll $S(1_2, 1_2, 2)$ already considered, and what was said there in regard to gauch-polygons applies equally well here. But the scroll now under consideration has no double generator, the plane quartic directing curve having only two double points, one on each double director, and this is the only quartic scroll on which the multiple curve is of order less than three.

2. *Proof of Theorem I.* — A plane through D cuts out two generators that meet in a point where the plane meets D', and if we revolve the plane about D, it will cut out, in succession, all the generators of the scroll, two at a time. The plane meets $C^{(a)}$ in a points, of which a definite number, say δ points, lie on D, and therefore there are $a - \delta$ points of $C^{(a)}$ on the two generators in the plane, taken together; for any given curve $C^{(a)}$, the number $a - \delta$ is a constant non-negative integer, say k. Let x and y be the number of points of $C^{(a)}$, respectively, on the two generators lying in a plane through D; then as the plane revolves about D, we always have $x + y = k$, and since x, y, and k are all non-negative integers and k is constant, there is only a finite number of values of x and of y that will satisfy this relation. Let us, for the moment, designate any generator by the number of points of $C^{(a)}$ on it, i. e. the generator x has x points of $C^{(a)}$ on it, etc. To any value of x, say g, there corresponds a certain value of y, say g', such that $g + g' = k$; if then a plane through D cuts out a generator g, it also cuts out a gen-

erator g'. As there is a finite number of pairs of values of x and y, we may arrange them in order of magnitude, calling g the greatest, i. e. we shall say that g is the greatest number of points of $C^{(a)}$ on any generator; correspondingly, g' is the smallest number of points of $C^{(a)}$ on any generator, since $g + g' = k$. Now a plane through D' cuts out two generators, and as we revolve the plane about D' the number of points of $C^{(a)}$ on the two generators in the plane is constant and equal to $a - \delta'$, where δ' is the number of points of $C^{(a)}$ on D'. If then we pass a plane through D' and a generator g', having the least number of points of $C^{(a)}$ on it, the other generator in this plane must have the greatest number g of points of $C^{(a)}$ on it; therefore $g + g' = a - \delta'$, and since $g + g' = a - \delta$, we have $\delta = \delta'$, i. e. $C^{(a)}$ meets each of the double directors, D and D', in the same number of points, δ. If now a plane be passed through a generator g and the director D it will cut out a generator g', and if through this generator g' and D' we pass a plane it will cut out another generator g, so that there are at least two generators g. Through these two generators g, the directors D and D' (which four lines form a gauch-quadrilateral), and an arbitrary point we can pass a quadric; D, D', and the two generators counting for $2 + 2 + 1 + 1 = 6$ lines in the intersection of the quadric and scroll. Each generator meets the quadric in two points, one on D and one on D', and if we take the arbitrary point on any generator, this generator will lie on the quadric, and the remaining intersection of the quadric and scroll will be another generator. Thus by varying this arbitrary point all the generators of the scroll, two at a time, will be successively cut out by a variable quadric that always contains D, D', and the two chosen generators g. This quadric always meets $C^{(a)}$ in $2a$ points, of which 2δ lie on D and D' and $2g$ lie on the two chosen generators g, so that the remaining $2a - 2\delta - 2g$ points lie on the other two generators cut out by the quadric; but $a - \delta = g + g'$ and therefore $2a - 2\delta - 2g = 2g'$. Now there is no generator that has fewer than g' points of $C^{(a)}$ on it and the number of points of $C^{(a)}$ on the two generators together is $2g'$; therefore each must have g' points of $C^{(a)}$ on it. Therefore every generator of the scroll, except the two chosen generators g, has g' points of $C^{(a)}$ on it. In like manner, if we choose two generators g', through which the variable quadric is always to pass, we have the sum of the number of points of $C^{(a)}$ on the other two generators cut out by the quadric always equal to $2a - 2\delta - 2g' = 2g$, and, since no generator has more than g points of $C^{(a)}$ on it, every generator on the scroll, except the two chosen ones, has g points of $C^{(a)}$ on it. Therefore $g = g'$ and

every generator meets $C^{(a)}$ in the same number of points, say α points. Two generators lie in a plane through D; therefore, $\delta = a - 2\,\alpha$, and

$\alpha \overline{\overline{<}} \frac{a}{2}$.

2. *Plane Curves.* — D and D' are both 2_1's. There is no system of conics on the scroll, for there is no double generator, and a plane through two generators cuts out D or D'.* A plane through one and only one generator cuts out a plane cubic, a 3_1, that meets each double director once and has no double point, since the section has only three double points which are the points where the generator in the plane meets the cubic, one on each of the double directors, and one where the plane is tangent to the scroll. An arbitrary plane cuts out a plane quartic, a 4_1, having two and only two double points, one on each double director. Every plane curve is, therefore, either the complete intersection of the plane and scroll, or else the residual consists entirely of generators, and consequently, by Theorems II and III, formula (1) holds for every plane curve on the scroll.

3. *Twisted Curves.* — It may be shown in exactly the same way as for the Quartic Scroll, $S\,(1_2, 1_2, 2)$, pp. 33–36, that formula (1) holds for every twisted curve on the scroll. It will be observed that, in the proof referred to, the double generator is shown to be a part of the residual; now, there is no double generator on this scroll, but disregarding the double generator the residual is still composed of curves of orders less than a, and the conclusion follows as before, without change.

4. The proof just employed is also applicable to the *Quartic Scroll with a two-fold* 2 (+ 2)*-tuple linear director and without a double generator*, $S\,(\overline{1_2, 1_2}, 4)$, (*Cayley's Fourth Species*), which is the limiting case of the scroll $S\,(1_2\; 1_2\; 4)$, just considered, where one of the double linear directors has moved up into coincidence with the other.

VI. Quartic Scroll, with a Double Conic and a Double Linear Director meeting it, $S\,(1_2, 2_2, 2)$. (Cayley's Seventh Species, $S\,(1, 2, 2)$.)

1. For convenience, let D represent the double linear director and let K represent the double conic. Any plane through D meets K in one more point, besides the intersection of K and D, and this is a double point on the section by the plane. The plane therefore cuts out D and a

* The section cannot consist of two proper conics. (See p. 25.)

conic having a double point on K, i. e. two lines that meet in this point, which are the two generators in the plane.

2. *Proof of Theorem I.* — The double curve on this scroll is a degenerate twisted cubic, consisting of the double conic K and the double director D. We can pass a quadric through nine arbitrary points, and if we take five of these on K and two on D, distinct from the point of intersection of K and D, K and D will both lie on the quadric and we shall still have two points at our disposal; now every generator meets K and D, and if we take one more point of the quadric on any generator A, it will lie on the quadric; the quadric will then intersect the scroll in K counted twice, D counted twice, and the generator A, and will, therefore, cut out one more generator. Making the quadric always contain K, D, and A, we have one point at our disposal, and by varying this point continuously we make the quadric cut out, in succession, all the generators of the scroll. $C^{(a)}$ meets K, D, and A in a definite number of points, and as it meets every quadric in $2a$ points it meets each generator in the same number of points, say α points. It is evident that A also meets $C^{(a)}$ in α points, for any other generator may be chosen as the one through which the quadric is always to pass, and then A will meet $C^{(a)}$ in the same number of points as the other generators, i. e. in α points. A plane through D cuts out two generators, and there are therefore $a - 2\alpha$ points of a_a on D. K is the complete intersection of its plane and the scroll, and there are therefore a points of a_a on K. The number of points of a_a on D cannot be less than zero, and therefore $\alpha \overline{\gtrless} \frac{a}{2}$.

3. *Plane Curves.* — The double director D meets every generator once and is therefore a 2_1. The double conic K is met once by each generator and is therefore a 4_1. The section by a plane not through D or K has three double points on the double curve, one on D and two on K. We have seen (p. 25), that the section cannot consist of two proper conics, and we know that a plane through two generators that meet on K, cuts out D, for each generator meets D; therefore, if a plane cuts out a simple conic, it cuts out also two generators that meet in a point on D, for there are no lines on the scroll but the generators, and D and two generators meet only on K or D; conversely, through every point of D pass two generators and their plane cuts out a proper conic; consequently, there is a system of conics that do not meet D, but meet K twice, for clearly the plane cannot meet D again, and the section cannot have a triple point on D unless the plane contains D; each of the two generators in the plane of any conic meets the conic twice, once

in a point where the plane meets K and once where the plane is tangent to the scroll; therefore the section has five double points and the plane is a double tangent plane. The conic and either generator lie on different sheets at the point where they meet K, and, regarding them as lying on the scroll, they do not intersect there. An arbitrary generator meets the plane once, and, since it does not meet either generator in the plane, it meets the conic once, and therefore every conic is a 2_1. Any plane through one and only one generator cuts out a plane cubic, a 3_1, having a double point on K and passing once through each of the points where the generator meets K and D. Any plane that does not contain D, K, or any generator, cuts out a plane quartic, a 4_1, having three double points, two on K and one on D.

The double conic K is the complete intersection of its plane with the scroll, and every plane quartic is the complete intersection of its plane with the scroll, and therefore formula (1) holds for K and for every plane quartic (Theorem II). A plane cubic is cut out by a plane through a single generator, D is cut out by a plane through two generators, and every conic is cut out by a plane through two generators, and therefore formula (1) holds for every plane cubic, for D, and for every conic (Theorem III). Formula (1) holds, therefore, for all plane curves on the scroll.

4. *Twisted Cubic*, 3_1. — Since $\alpha \leqq \frac{a}{2}$, every twisted cubic is a 3_1. We have seen that there are a points of a_α on K, i. e. there are three points of the twisted cubic on K, and, since we have two points at our disposal in the determination of the quadric that cuts out the twisted cubic, we can take these two points on K and thus make the quadric contain K. The residual, which is of order 5, will then consist of K, which counts for 4, and one generator, and therefore formula (1) holds for every twisted cubic on the scroll (Theorem III).

5. *Twisted Quartics*, 4_2 *and* 4_1. — Since $\alpha \leqq \frac{a}{2}$, every twisted quartic is either a 4_2 or a 4_1. A 4_2 may be cut out by a quadric; every generator will then meet the quadric twice on the quartic curve 4_2 and cannot meet it again without lying on it; consequently, every generator that has on it a point of the residual must lie on the quadric and form part of the residual, and the residual therefore consists of four generators; therefore formula (1) holds for every 4_2 (Theorem III). For the 4_1, we take $\nu = 3$; then $r = 8$, and we have $19 - 13 = 6$ points at our disposal in the determination of the cubic surface $S^{(3)}$ that cuts out the 4_1. There

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are four points of the twisted quartic 4_1 on K, and if we take three more points of $S^{(3)}$ on K, $S^{(3)}$ will contain K; D meets $S^{(3)}$ twice on the curve 4_1, and if we take two more points of $S^{(3)}$ on D, $S^{(3)}$ will contain D; every generator will then meet $S^{(3)}$ once on D, once on K, and once on the curve 4_1, and as we still have one point at our disposal we can make $S^{(3)}$ contain a generator; this generator, D, and K count for 7 in the order of the residual, and therefore $S^{(3)}$ cuts out one more generator. The residual then consists of K, D, and two generators, and therefore formula (1) holds for the twisted quartic 4_1 (Theorem III). Formula (1) holds, therefore, for every twisted quartic.

6. *Twisted Curves in General.* — When a is odd we take $\nu = \frac{a+1}{2}$; then $r = a + 2$, and, by formula (2), we have $\frac{a+1}{2}$ points at our disposal in the determination of $S^{(\nu)}$. If K does not lie on $S^{(\nu)}$, it meets $S^{(\nu)}$ in $2\left(\frac{a+1}{2}\right) = a + 1$ points; but we have seen that there are a points of a_α on K, and consequently, if we take two more points of $S^{(\nu)}$ on K, $S^{(\nu)}$ will contain K; this we can always do, since the number of points at our disposal is $\frac{a+1}{2} \gtreqless 2$ for $a \gtreqless 3$. The residual will then consist of K and a curve of order $r - 4 = a - 2$.

When a is even, we distinguish two kinds of curves according as $\frac{a}{2}$ is odd or even. If $\frac{a}{2}$ is odd we take $\nu = \frac{a}{2}$; then $r = a$, and if δ be the number of points of a_α on D, we have seen that $\delta = a - 2\alpha$: now, if $\delta \geqq \frac{a}{2} + 1$, i.e. if $\alpha \leqq \frac{a-2}{4}$, D meets $S^{(\nu)}$ in at least $\frac{a}{2} + 1 = \nu + 1$ points,* and therefore lies on $S^{(\nu)}$, so that the residual consists of D and a curve of order $r - 2 = a - 2$; if $\alpha > \frac{a-2}{4}$, the residual either breaks up into curves of orders less than a, or else it is a curve of order a, say an a_ρ, where $\rho = \frac{a}{2} - \alpha$, since each generator meets $S^{(\nu)}$ in $\nu = \frac{a}{2}$ points, of which α lie on a_α and ρ on a_ρ; then, since $\alpha > \frac{a-2}{4}$, $\rho < \frac{a+2}{4}$; but ρ is an integer and $\frac{a+2}{4}$ is an integer, so that $\rho \leqq \frac{a+2}{4} - 1$, i.e. $\rho \leqq \frac{a-2}{4}$, and a_ρ is therefore one of the curves

* Or $S^{(\nu)}$ can be made to pass through $\nu + 1$ points of D. (See p. 25.)

just considered, that can be cut out by such an $S^{(\nu)}$ that the residual will consist of D and a curve of order $a-2$. If $\frac{a}{2}$ is even and $\nu=\frac{a}{2}$, the residual is of order a; then δ must be even, being equal to $a-2\alpha$, and, if $\delta \geqq \frac{a}{2}+2$, i.e. if $\alpha \leqq \frac{a-4}{4}$, D lies on $S^{(\nu)}$, and the residual consists of D and a curve of order $a-2$. If $\alpha > \frac{a-4}{4}$, then $\alpha \geqq \frac{a}{4}$. When $\alpha > \frac{a}{4}$ the residual either breaks up into curves of orders less than a, or else it is a curve of order a, say an a_ρ, where $\rho=\nu-\alpha<\frac{a}{4}$, i.e. $\rho \leqq \frac{a-4}{4}$, and consequently a_ρ is a curve like that just considered, that can be cut out by an $S^{(\nu)}$ such that the residual will consist of D and a curve of order $a-2$. Finally, when $\alpha=\frac{a}{4}$, we take $\nu=\frac{a}{2}+1$; then $r=a+4$, and, by formula (2), we have $a+2$ points at our disposal in the determination of $S^{(\nu)}$. If K does not lie on $S^{(\nu)}$, it meets $S^{(\nu)}$ in $a+2$ points; now K has a points of a_α on it, and if we make $S^{(\nu)}$ pass through three more points of K, not on a_α, $S^{(\nu)}$ will contain K; this we can always do and still have at least three more points at our disposal, since $a+2 \geqq 6$ for $a \geqq 4$. Since $\alpha=\frac{a}{4}$, $\delta=\frac{a}{2}$, and if we make $S^{(\nu)}$ pass through two more points of D, not on a_α, $S^{(\nu)}$ will cut out D. The residual will then consist of K, D, and a curve of order $r-4-2=a-2$.

The twisted curves on this scroll can therefore be divided into two groups, viz., group (1), those curves of order a, each of which can be cut out by such an $S^{(\nu)}$ that the residual will consist of curves of orders less than a, and group (2), those curves of order a, each of which can be cut out by such an $S^{(\nu)}$ that the residual will be a curve of order a and of group (1). Therefore, by the same reasoning as that employed for Quartic Scroll $S(1_2, 1_2, 2)$, p. 36, Formula (1) holds for all curves on the scroll.

VII. Quartic Scroll, with a Double Twisted Cubic met twice by each Generator and with a Simple Linear Director, $S(3_2^2, 1)$. (Cayley's Eighth Species, $S(1, 3^2)$.)

1. Let Q represent the twisted cubic, which is the double curve on the scroll. Through each point of Q pass two generators; the plane of these two generators contains the linear director, since each generator

meets the linear director once, and therefore this plane cuts out also a third generator, i.e. any plane through the linear director meets Q in three points, say L, M, and N, and cuts out the three generators LM, MN, and NL.

2. *Proof of Theorem I.* — We can make a quadric pass through Q by making it pass through seven points of Q; and since Q is a double cubic it counts for six in the order of the complete intersection of the quadric and scroll; if the eighth point for the determination of the quadric be taken on any generator A, the quadric will contain A, since each generator meets Q twice; the remaining intersection will be any generator on which we choose to take the ninth point for the determination of the quadric, and if we keep the first eight points fixed and vary the ninth point continuously, the quadric will cut out in succession the different generators of the scroll. A fixed number of points of $C^{(a)}$ lie on Q and the chosen generator A, and therefore every generator contains the same number of points of $C^{(a)}$, say α points. Any generator, other than A, can be chosen, through which the quadric is always to pass, and therefore there are α points of $C^{(a)}$ on A. Since the quadric meets a_α in $2\,a$ points, there are $2\,a - 2\,\alpha = 2\,(a - \alpha)$ points of a_α on Q. Three generators lie in a plane through the linear director; therefore $\alpha \overline{\underline{<}} \frac{a}{3}$, and there are $a - 3\,\alpha$ points of a_α on the linear director.

3. *Plane Curves.* — Since the section by a plane cannot consist of two proper conics (p. 25), a plane through a proper conic would either cut out two generators or the simple director and one generator; but we have seen that a plane through two generators or through the simple linear director cuts out three generators and the linear director; therefore there can be no proper conic on the scroll.

A plane through one, and only one, generator cuts out a plane cubic, a 3_1, having a double point on Q and passing once through each of the two points where the generator meets Q; the generator meets the plane cubic in one other point where the plane is tangent to the scroll. An arbitrary plane cuts out a plane quartic, a 4_1, having three double points on Q, and since any plane quartic is the complete intersection of its plane with the scroll, formula (1) holds for it (Theorem II). The simple linear director, a 1_1, is cut out by a plane through three generators, and every plane cubic is cut out by a plane through one generator, and therefore formula (1) holds for the simple linear director and for every plane cubic (Theorem III). Therefore formula (1) holds for all plane curves on the scroll.

The double cubic Q, although not a plane curve, will be considered here. We have seen that it is cut out by a quadric through two generators, and therefore formula (1) holds for it (Theorem III). It is met twice by every generator, and is, therefore, a 6_2. A plane quartic has a branch on each sheet at each of the three points where it meets Q, and the number of its intersections with Q is $6, (6_2, 4_1) = 6 + 8 - 8 = 6$. A plane cubic meets Q four times, twice at the double point of the plane cubic and once at each of the other two points where the plane meets $Q, (6_2, 3_1) = 6 + 6 - 8 = 4$. The linear director does not meet Q, $(6_2, 1_1) = 6 + 2 - 8 = 0$.

4. *Twisted Cubic*, 3_1. — Since $a \overline{\gtreqless} \frac{a}{3}$, every twisted cubic is a 3_1. We take $\nu = 3$; then $r = 9$, and we have $19 - 10 = 9$ points at our disposal in the determination of $S^{(3)}$ that cuts out the twisted cubic. The number of points of the twisted cubic on Q is $2(a - a) = 4$, and if we take 6 more points of $S^{(3)}$ on Q, $S^{(3)}$ will contain Q; this leaves $9 - 6 = 3$ points at our disposal, and, since each generator now meets $S^{(3)}$ twice on Q and once on the twisted cubic, we can take one more point of $S^{(3)}$ on each of three generators and $S^{(3)}$ will then contain those three generators; the residual will then consist of Q and three generators, and therefore formula (1) holds for every twisted cubic (Theorem III). Since three generators lie in a plane through the linear director, a twisted cubic does not meet the linear director, $(3_1, 1_1) = 3 + 1 - 4 = 0$.

5. *Twisted Quartic* 4_1. — Since $a \overline{\gtreqless} \frac{a}{3}$, every twisted quartic is a 4_1. We take $\nu = 3$; then $r = 8$, and the number of points at our disposal in the determination of $S^{(3)}$ is $19 - 13 = 6$. There are $2(a - a) = 6$ points of the twisted quartic on Q, and if we take four more points of $S^{(3)}$ on Q, $S^{(3)}$ will contain Q. Each generator will then meet $S^{(3)}$ twice on Q, and once on the twisted quartic, and, since we still have two points at our disposal, we can make $S^{(3)}$ cut out two generators. The residual will then consist of Q and two generators, and therefore formula (1) holds for every twisted quartic.

6. *Twisted Curves in General.* — When a is odd we take $\nu = \frac{a+3}{2}$; then $r = a + 6$, and by formula (2) we have $\frac{3a+9}{2}$ points at our disposal in the determination of $S^{(\nu)}$. If δ be the number of points of a_a on Q, $\delta = 2(a - a) \overline{\gtreqless} \frac{4a}{3}$, since $a \overline{\lesseqgtr} \frac{a}{3}$. That $S^{(\nu)}$ may contain Q, it must

pass through $3\nu + 1 = \frac{3a + 11}{2}$ points of Q, and therefore, if we take $\frac{3a + 11}{2} - \delta \overline{\gtrless} \frac{a + 33}{6}$ of the points at our disposal on Q, $S^{(\nu)}$ will contain Q; if λ be the number of points left at our disposal, after making $S^{(\nu)}$ contain Q, $\lambda \overline{\gtrless} \frac{3a + 9}{2} - \frac{a + 33}{6}$, i. e. $\lambda \overline{\gtrless} \frac{4a}{3} - 1$. Now each generator meets $S^{(\nu)}$ twice on Q and a times on a_a, i. e. at least three times since $a \overline{\gtrless} 1$; if, then, we make $S^{(\nu)}$ pass through $\frac{a + 3}{2} + 1 - 3 = \frac{a - 1}{2}$ other points on any generator, that generator will lie on $S^{(\nu)}$. We can, therefore, make at least two generators lie on $S^{(\nu)}$, since $\frac{4a}{3} - 1 > 2\left(\frac{a - 1}{2}\right)$, and the residual will then consist of Q, two generators, and a curve of order $r - 6 - 2 = a - 2$.

When a is even we take $\nu = \frac{a}{2} + 1$; then $r = a + 4$, and, by formula (2), we have $a + 2$ points at our disposal in the determination of $S^{(\nu)}$. The number of points of a_a on Q is $\delta = 2(a - a) \overline{\gtrless} \frac{4a}{3}$. In order for $S^{(\nu)}$ to contain Q, it must pass through $3\nu + 1 = \frac{3a + 8}{2}$ points of Q, and therefore in addition to the δ points we must make $S^{(\nu)}$ pass through $\frac{3a + 8}{2} - \delta \overline{\gtrless} \frac{a + 24}{6}$ other points of Q; this we can always do, since the number of points at our disposal is $a + 2 > \frac{a + 24}{6}$ for $a \overline{\gtrless} 4$. The residual will then consist of Q and a curve of order $r - 6 = a - 2$.

We have shown, then, that every twisted curve of order a can be cut out by an $S^{(\nu)}$ such that the residual is composed of curves of orders less than a; we have also shown that formula (1) holds for every plane curve and for every twisted curve of order 3 or 4; it holds, therefore, for every twisted curve of order 5, and therefore for every twisted curve of order 6, and so on (Theorem III); therefore formula (1) holds for every curve on the scroll.

VIII. Quartic Scroll, with a Double Twisted Cubic met Twice by each Generator, $S(3_2^2, 2)$. (Cayley's Tenth Species, $S(3^2)$.)

1. Let Q be the double twisted cubic. Through every point of Q pass two generators. The scroll differs from the Quartic Scroll $S(3_2^2, 1)$

in not having a simple linear director, in consequence of which a plane through two generators cuts out a proper conic.

2. *Proof of Theorem I.* — By passing a quadric surface through Q and any chosen generator, Theorem I is proved in exactly the same way as it is for the Quartic Scroll $S(3_2^2, 1)$, p. 44., and if α be the number of points of $C^{(a)}$ on each generator, we see, as before, that the number of points of a_α on Q is $2(a - \alpha)$. Since two generators lie in a plane, $\alpha \overline{\overline{<}} \frac{a}{2}$.

3. *Plane Curves.* — Since there are no lines on the scroll except the generators, a plane through two generators cuts out a proper conic, and since the section has three double points on Q, the conic passes through each of the two points in which the generators meet Q, different from their point of intersection. Each conic is a 2_1, and regarding the conic and generators as lying on the scroll, the conic is met once only by each generator in its plane, since at the point where the conic and generator cross Q they lie on different sheets. The two points where the two generators meet the conic, not on Q, are points of tangency of the plane, and therefore through every point of Q there is a double tangent plane to the scroll. By Theorem III, formula (1) holds for every conic. There is no 1_1 on the scroll, and the other plane curves, the cubic, 3_1, and the quartic, 4_1, are the same as those of the same order on the Quartic Scroll $S(3_2^2, 1)$ and therefore formula (1) holds for all curves on the scroll. Since Q is a double twisted cubic met twice by each generator, it is a 6_2, and since it is cut out by a quadric through two generators, formula (1) holds for it (Theorem III). We have seen that Q meets each conic twice, and the formula gives

$$(6_2, 2_1) = 6 + 4 - 8 = 2.$$

4. *Twisted Curves.* — It is proved in exactly the same way as for the Quartic Scroll $S(3_2^2, 1)$ that formula (1) holds for every twisted curve on the scroll; therefore, it holds for every curve on the scroll.

IX. Quartic Developable.

1. This surface is formed by the tangents to a twisted cubic, E; E is then a double curve on the surface, and from one point of view, this surface is the limiting case of the Quartic Scroll $S(3_2^2, 1)$ where the two points in which any generator meets the twisted cubic have become consecutive; the twisted cubic then becomes the "edge of regression" and the three double points of the section by a plane become cusps.

2. *Proof of Theorem I.* — By passing a quadric through E, which is the cuspidal edge of the developable, and any chosen generator, Theorem I is proved in exactly the same way as it is for the Quartic Scroll $S(3_2^2, 1)$, p. 44, and if α be the number of points of $C^{(a)}$ on each generator, E has $2(a - \alpha)$ points of a_α on it. Since E is cut out by a quadric through two generators, formula (1) holds for it (Theorem III). Being a double twisted cubic met twice by each generator, E is a 6_2, and formula (1) gives, for the number of intersections of E with a_α,

$$(6_2, a_\alpha) = 6\alpha + 2a - 8\alpha = 2(a - \alpha).$$

3. *Plane Curves.* — A "plane of the system," i. e. an osculating plane of E, cuts out two consecutive generators and a proper conic.* This conic, which is tangent to E,† can never break up, for there are no lines on the surface except the generators, and a plane cannot contain more than two generators (which must, moreover, be consecutive), since it cannot meet E in more than three points. Each conic is a 2_1, and there is a conic through every point of E.

A plane through one and only one generator cuts out a plane cubic, a 3_1, that has a cusp at the point not on the generator, where the plane meets E, and has the generator for its real inflectional tangent at the point where the generator meets E; for, since the two consecutive points in which the generator meets E must be double points of the section of a plane through the generator, the plane cubic must pass through each of these points, i. e. it meets the generator in at least two points there; but the generator at this point crosses from one sheet to the other, say from the upper to the lower, while the plane cubic crosses from the lower to the upper; therefore they cross each other and intersect in an odd number of points, and consequently they intersect in three points, i. e. the generator is the real inflectional tangent.

Every plane quartic has three cusps, one at each of the three points where its plane meets E, and since it is the complete intersection of its plane with the developable, formula (1) holds for it (Theorem II).

Every conic is cut out by a plane through two generators and every plane cubic is cut out by a plane through one generator, and therefore formula (1) holds for every conic and for every plane cubic (Theorem III). Formula (1) holds, therefore, for every plane curve on the developable.

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* Salmon's Geom. of Three Dimensions, § 339.

† V. Jamet, C. R., 1885, Vol. 100, pp. 1332–35.

4. *Twisted Curves.* — It may be shown in exactly the same way as for the Quartic Scroll $S(3_2^2, 1)$, that formula (1) holds for every twisted curve on the developable.

Formula (1) holds, therefore, for every curve on the developable.

X. Quartic Cones.

1. Using the plane quartic curves as a base, we may say that there are ten different species of quartic cones, corresponding to the ten species of plane quartic curves.* But for our present purpose we need not distinguish between double edges and cuspidal edges, and it will be convenient to divide the cones into five groups, viz., (I) cones having a triple edge, (II) cones having three edges double or cuspidal, (III) cones having two edges double or cuspidal, (IV) cones with one double or cuspidal edge, and (V) non-singular cones, i. e. cones with no multiple edges.†

A curve $C^{(a)}$, lying on a cone, has a k-tuple point at the vertex, where $0 \gtreqless k$, and any plane through the vertex meets $C^{(a)}$ in k points there.

2. *Theorem I.* — Every edge of a quartic cone meets $C^{(a)}$ in the same number of points, say α points (in addition to the k points at the vertex); the number of points of $C^{(a)}$ on a double edge is $2\,\alpha$, and the number of points of $C^{(a)}$ on a triple edge is $3\,\alpha$.

Proof. — Considering group (I), if we pass a plane through the triple edge and revolve it about this triple edge, it will cut out, in succession, all the edges of the cone; the plane meets $C^{(a)}$ in k points at the vertex and in a fixed number of points, say τ points, on the triple edge, and, as it meets $C^{(a)}$ altogether in a points, the same number of points, say α points, lie on each edge of the cone. An arbitrary plane through the vertex meets the cone in four edges, and therefore

$$4\,\alpha + k = a, \qquad \text{i. e. } 4\,\alpha = a - k.$$

Taking the plane through the triple line, we have

$$\alpha + \tau = a - k = 4\,\alpha, \qquad \text{i. e. } \tau = 3\,\alpha.$$

Consider now group (II), and pass a quadric cone through the three double or cuspidal edges and any chosen edge A of the quartic cone; these count for seven edges in the intersection of the two cones, and

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* Salmon's Higher Plane Curves, § 243.

† When not qualified by the words *double, cuspidal, triple, multiple,* etc., the word *edge* will always mean a *simple edge* of the cone.

they must have one more edge in common; if, then, we vary continuously the fifth arbitrary edge that determines the quadric cone, this cone will always pass through the three double or cuspidal edges and the edge A of the quartic cone, and will cut out all the edges of the quartic cone one at a time. Every quadric cone meets $C^{(a)}$ in $2a$ points, of which a definite number lie on the three double or cuspidal edges and the edge A, and therefore the same number of points of $C^{(a)}$, say α points, lie on each edge of the quartic cone; it is evident that, if any other edge be chosen through which the quadric cone is always to pass, the edge A will have α points of $C^{(a)}$ on it. If we pass an arbitrary plane through any of the double or cuspidal edges, it will also cut out two edges of the quartic cone, and, if δ be the number of points of $C^{(a)}$ on this double or cuspidal edge, we have

$$\delta + 2\alpha = a - k = 4\alpha, \qquad \text{i. e. } \delta = 2\alpha,$$

since an arbitrary plane through the vertex cuts out four edges, giving $4\alpha = a - k$.

To prove the theorem for group (III), we may employ a method analogous to that used for the Quartic Scroll $S(1_2, 1_2, 4)$, p. 37; corresponding to the planes there, through the double directors, we should use here the planes through the double or cuspidal edges, and instead of the quadric used there we should now employ a quadric cone.

We shall not consider this proof in detail, but shall now give a proof of the theorem for groups (IV) and (V), — a proof which holds for the other three groups as well, and therefore for all quartic cones.

If p is the number of points of $C^{(a)}$ on any edge of the cone, it is evident that $0 \overline{\gtrless} p \overline{\gtrless} a - 1$, and p can have, at most, a different values. For convenience, we shall say that all the edges for which p has the same value belong to the same set; and if there is an infinite number of edges in a set, we shall call it an infinite set, or, if there is a finite number, a finite set. There are, at most, a different sets, and, since a is finite, at least one of these must be an infinite set. Let us suppose, first, that there is only one infinite set, and let α be the value of p for this infinite set; we can then pass a plane through one of these α-edges (i. e. an edge having α points of $C^{(a)}$ on it) and turn it about so that it shall not contain any edge of any finite set; this plane will then cut out four α-edges, and as it meets $C^{(a)}$ in a points, of which k lie at the vertex, we have

$$4\alpha + k = a, \qquad \text{i. e. } a - k = 4\alpha.$$

Now we can pass a plane through any edge of any of the finite sets and

turn it about so that it will not contain any other edge of any finite set, and this plane will then cut out three α-edges in addition to the chosen edge; if β is the number of points of $C^{(a)}$ on the chosen edge, we have

$$\beta + 3\,a = a - k = 4\,a, \qquad \text{i. e. } \beta = a.$$

Therefore every edge of the cone has a points of $C^{(a)}$ on it. If the cone has a double or cuspidal edge, an arbitrary plane through it will cut out two α-edges, and if δ is the number of points of $C^{(a)}$ on the double or cuspidal edge

$$\delta + 2\,a = a - k = 4\,a, \qquad \text{i. e. } \delta = 2\,a.$$

If the cone has a triple edge, having τ points of $C^{(a)}$ on it, an arbitrary plane through this triple edge will cut out one α-edge, and we have

$$\tau + a = a - k = 4\,a, \qquad \text{i. e. } \tau = 3\,a.$$

Therefore, the theorem holds when there is only one infinite set. Suppose now that any number of the a sets are infinite, and let a be the least value of p belonging to any of these infinite sets. Pass a cubic cone through nine of these α-edges; then, since the cubic cone meets $C^{(a)}$ in $3\,a$ points, of which $3\,k$ lie at the vertex and $9\,a$ on the nine chosen edges, the remaining $3\,a - 3\,k - 9\,a$ points lie on the three other edges in which the cubic and quartic cones intersect, and at least one of these three edges must have as many as $a - k - 3\,a$ points of $C^{(a)}$ on it. Keeping seven of the α-edges fixed, we can vary the other two in such a way that the cubic cone, determined each time by the nine α-edges, will have for its remaining intersection with the quartic cone three edges different from those of any other such cone previously determined; since there is an infinite number of α-edges, we get, in this way, an infinite number of such cubic cones, and therefore an infinite number of edges each having as many as $a - k - 3\,a$ points of $C^{(a)}$ on it; having an infinite number of such edges, we can so choose two of them that their plane will not pass through any edge of any finite set, because there is a finite number of edges in all the finite sets taken together; besides the chosen edges this plane will then cut out two edges belonging to the infinite sets. Now the number of points of $C^{(a)}$ on the two chosen edges taken together is equal to or greater than $2\,a - 2\,k - 6\,a$, and if β and γ be the number of points of $C^{(a)}$, respectively, on the other two edges in the plane we must have

$$2\,a - 2\,k - 6\,a + \beta + \gamma \mathrel{\overline{\gtrless}} a - k,$$

and, since neither β nor γ can be less than a,

$$2\,a - 2\,k - 6\,a + 2\,a \mathrel{\overline{\gtrless}} a - k,$$

i. e. $$a - k \gtreqless 4\,\alpha.$$

If, now, we pass a plane through any edge of any infinite set and turn it about so that it will not pass through any edge of any finite set, it will cut out three other edges of the infinite sets. Let p_1, p_2, p_3, and p_4 be the number of points of $C^{(a)}$, respectively, on the four edges in this plane; we have then

$$p_1 + p_2 + p_3 + p_4 = a - k \gtreqless 4\,\alpha,$$

and, since neither p_1, p_2, p_3, nor p_4 can be less than α, each must be equal to α and $a - k = 4\,\alpha$. Therefore, every edge of every infinite set meets $C^{(a)}$ in α points, i. e. there is only one infinitive set, and we have proved that the theorem holds in this case.

3. On the cones, a curve $C^{(a)}$ is, as before, designated by the symbol a_α, but α now means the number of points, other than those at the vertex, in which an arbitrary edge of the cone meets the curve $C^{(a)}$. We shall now show that formula (1), $(a_\alpha, b_\beta) = a\,\beta + b\,\alpha - 4\,\alpha\,\beta$, gives, for the quartic cones, the number of intersections of the two curves, a_α and b_β, exclusive of the number of their intersections at the vertex of the cone.

We have seen that $a - k = 4\,\alpha$, i. e. $k = a - 4\,\alpha$, where k is the number of branches of a_α through the vertex, i. e. a_α has an $(a - 4\,\alpha)$-tuple point at the vertex. Let a_α be the complete intersection of $S^{(\nu)}$ and the quartic cone, and let b_β be an arbitrary curve on the cone; the total number of intersections of a_α and b_β is the number of intersections of $S^{(\nu)}$ and b_β, which is $b\,\nu$, and, since $a = 4\,\nu$, $b\,\nu = \frac{a\,l}{4}$; now, at the vertex, $S^{(\nu)}$ has a point of multiplicity $\frac{a - 4\,\alpha}{4}$ (since a_α is the complete intersection of $S^{(\nu)}$ and the cone and has an $(a - 4\,\alpha)$-point at the vertex), and b_β has a $(b - 4\,\beta)$-tuple point, so that b_β meets $S^{(\nu)}$ in

$$\left(\frac{a - 4\,\alpha}{4}\right)(b - 4\,\beta) = \frac{a\,b}{4} - (a\,\beta + b\,\alpha - 4\,\alpha\,\beta)$$

points at the vertex; since the total number of intersections of b_β and $S^{(\nu)}$ is $\frac{a\,b}{4}$, the number of their intersections exclusive of those at the vertex, i. e. the number of intersections of b_β and a_α exclusive of those at the vertex, is $\frac{a\,b}{4} - \frac{a\,b}{4} + (a\,\beta + b\,\alpha - 4\,\alpha\,\beta) = a\,\beta + b\,\alpha - 4\,\alpha\,\beta$, which is the number given by formula (1).

Since the formula holds when a_α is the complete intersection of the cone and $S^{(\nu)}$, by Theorem III, it holds when a_α is the partial intersec-

tion of the cone and $S^{(\nu)}$, provided we can always cut out a_a by an $S^{(\nu)}$ such that the residual is of order less than a or breaks up into components, each of which is of order less than a; for we know that it holds for the number of intersections of any edge, a 1_0, with an arbitrary curve b_β, giving $(1_0, b_\beta) = \beta$. Each edge is a 1_0, each double or cuspidal edge is a 2_0, and each triple edge is a 3_0, since the edges and multiple edges meet only at the vertex. Every multiple edge can be cut out by a plane such that the residual will consist entirely of generators, and therefore formula (1) holds for every multiple edge (Theorem III). Since the plane quartic cannot break up without the cone breaking up (unless it consists entirely of edges or multiple edges), the only plane curve, besides the edges and multiple edges, that can lie on the cone is a plane quartic which is the complete intersection of its plane with the cone, and therefore formula (1) holds for every plane curve.

4. *Twisted Curves.* — There is no cubic curve on a quartic cone, since $1 \overline{\overline{<}} a$ and $4\alpha \overline{\overline{<}} a$, and, therefore, $4 \overline{\overline{<}} a$.

Every twisted quartic is a 4_1, since $\alpha \overline{\overline{<}} \frac{a}{4}$, and, since $k = a - 4\alpha = 0$, it does not go through the vertex. Any twisted quartic can be cut out by a cubic monoid,* for we can pass the monoid through $19 - 4 = 15$ arbitrary points, and if we take thirteen of these on the quartic curve, the monoid will contain it. The node of the monoid is taken at the vertex of the cone; each edge of the cone then meets the cubic monoid twice at the vertex and once on the quartic curve, and cannot meet it again without lying on it; therefore the monoid cannot meet the cone in any other curve, and the residual consists entirely of edges or multiple edges of the cone.

Every twisted quintic is a 5_1, and has one branch through the vertex, $k = 5 - 4 = 1$; it can be cut out by a cubic monoid which can be passed through 15 arbitrary points; for the quintic meets the monoid twice at the vertex, and if we pass the monoid through 14 other points of the quintic, it will contain the quintic. Every edge of the cone meets the monoid twice at the vertex and once on the quintic curve, and therefore the residual consists entirely of edges or multiple edges of the cone.

Every twisted sextic is a 6_1, and has two branches through the vertex, since $k = 6 - 4 = 2$; it can be cut out by a cubic monoid whose node is at the vertex of the cone; for the sextic meets the monoid four times

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* A *monoid* of order m is a surface of order m having an $(m-1)$-tuple point. (Cayley.)

at the vertex, and if we pass the monoid through 15 other points of the sextic it will contain the sextic. The residual will then consist entirely of edges or multiple edges of the cone.

In the same way it may be shown that all curves of order 7, and all curves of orders 8, 9, and 10 for which $\alpha = 1$, can be cut out by a quartic monoid such that the residual will consist of edges or multiple edges of the cone.

When $\alpha \gtreqless 2$ (for which $a \gtreqless 8$), or when $a \gtreqless 11$, i.e. for all curves not yet considered, we must take $\nu \gtreqless 5$, where ν is the order of the surface of lowest order that cuts out a_α such that the residual consists of edges or multiple edges of the cone. We shall now show that such a surface can be found for any a_α, and the value of ν given in terms of a and α. Since the residual is to consist entirely of edges or multiple edges of the cone, an arbitrary edge must meet the required surface in $\nu - \alpha$ points at the vertex, i.e. the surface must have a $(\nu - \alpha)$-tuple point at the vertex of the cone. Let $M^{(\nu)}_{\nu-\alpha}$ be the required surface. When $\nu > 4$ we must take care that $M^{(\nu)}_{\nu-\alpha}$ does not break up into the quartic cone and a surface which must be an $M^{(\nu-4)}_{\nu-\alpha-4}$, i.e. a surface which has a $(\nu - \alpha - 4)$-tuple point at the vertex; $M^{(\nu-4)}_{\nu-\alpha-4}$ can pass through only

$$\tfrac{1}{6}[(\nu-3)(\nu-2)(\nu-1) - (\nu-\alpha-4)(\nu-\alpha-3)(\nu-\alpha-2)] - 1$$

arbitrary points, different from the vertex, and consequently if we make $M^{(\nu)}_{\nu-\alpha}$ pass through

$$\text{(A)} \quad \tfrac{1}{6}[(\nu-3)(\nu-2)(\nu-1) - (\nu-\alpha-4)(\nu-\alpha-3)(\nu-\alpha-2)]$$

arbitrary points, not on the quartic cone, it cannot have the quartic cone as a component; the number of arbitrary points remaining, to determine $M^{(\nu)}_{\nu-\alpha}$, must be great enough to make it contain a_α, which has $a - 4\alpha$ branches through the vertex; consequently we have

$$(3) \ldots \quad a\nu + 1 - (a - 4\alpha)(\nu - \alpha) \lesseqgtr \tfrac{1}{6}[(\nu+1)(\nu+2)(\nu+3) - (\nu-\alpha)(\nu-\alpha+1)(\nu-\alpha+2) - (\nu-3)(\nu-2)(\nu-1) + (\nu-\alpha+4)(\nu-\alpha-3)(\nu-\alpha-2)] - 1,$$

from which we obtain the relation

$$1 + a\alpha + 4\alpha\nu - 4\alpha^2 \lesseqgtr 4\alpha\nu - 2\alpha^2 + 4\nu - 4\alpha - 3,$$

i.e.

$$(4) \ldots \quad \nu = \frac{\alpha(a - 2\alpha + 4) + 4}{4},$$

or the next greater integer. When $\nu = 3$ or 4 we saw that $\alpha = 1$, so that the expression (A) that enters into eq. (3) vanishes for $\nu = 3$, as it should, and is equal to unity for $\nu = 4$; therefore eq. (4) gives the correct value of ν for all twisted curves on the cone. If then we take the value of ν given by eq. (4), any twisted curve a_a can be cut out by an $M^{(\nu)}_{\nu-a}$ such that the residual will consist entirely of edges or multiple edges of the cone, and therefore formula (1) holds for all twisted curves on the quartic cones. In determining the above value of ν, no account was taken of the actual multiple points, not at the vertex, that a_a may have. We have seen (p. 24) that an m-tuple point reduces by $m - 1$ the number of points of a_a through which it is necessary to make $M^{(\nu)}_{\nu-a}$ pass in order for it to contain a_a, if this m-tuple point be taken as one of them; therefore, if $\alpha\ (a - 2\,\alpha + 4) + 4 \equiv 1$ (mod. 4) and a_a has a double point not at the vertex, the value of ν may be taken one less than that given by eq. (4); if $\alpha\ (a - 2\,\alpha + 4) + 4 \equiv 2$ (mod. 4) and a_a has two double points or one triple point, not at the vertex, the value of ν from eq. (4) is reduced by unity, and so on. If a_a has four double points, or two triple points, or two double points and a triple point, or a double point and a 4-tuple point, or one 5-tuple, not at the vertex, then the value of ν is always one less than that given by eq. (4).

We have shown, then, that formula (1) gives the number of intersections, aside from those at the vertex, of any two curves on any quartic cone. It is to be observed that any branch of a_a through the vertex has an edge of the cone as its tangent at that point and that one of the two consecutive points, in which the edge meets the branch there, is one of the α points of a_a that lie on this edge; if a double or cuspidal edge is tangent to a branch of a_a at the vertex, then two of the $2\,\alpha$ points of a_a that lie on it are consecutive to the vertex; and if a triple edge is tangent to a branch of a_a at the vertex, then three of the $3\,\alpha$ points of a_a that lie on it are consecutive to the vertex. If, therefore, a_a and b_β each have a branch through the vertex tangent to the same edge, i. e. a branch of a_a tangent to a branch of b_β at the vertex, then one of these two intersections of the curves at this point is included in the number of intersections given by formula (1). In like manner, if a_a and b_β each have a branch through the vertex, and these branches have there a common inflectional or cuspidal tangent or have any peculiar relation to one another, the excess of the number of intersections that these branches have there over the number of intersections that two arbitrary branches would have there is included in the number of intersections given by formula (1).

5. We shall now consider the twisted quartics more in detail. Every twisted quartic is a 4_1, and, since it has at least two apparent double points, the cone on which it lies must have at least two double or cuspidal edges, i. e. no twisted quartic can lie on the cones of groups (IV) and (V); moreover, the cubic monoid, which cuts out the twisted quartic, has six * lines on it through the vertex, and these six lines must count for eight, the order of the residual intersection of the cone and monoid, and therefore the cone must have, at least, two double or cuspidal edges with which two of the six lines coincide. Through a "quartic of the first kind" can be passed an infinity of quadrics, i. e. we can pass a quadric through a "quartic of the first kind" and any arbitrary point; let this quadric be passed through the vertex. The "quartic of the first kind" has two apparent double points, and the two double edges, of the cone, on which they lie, meet the quadric twice on the curve and once at the vertex, and therefore lie entirely on it; these double edges are, therefore, the two generators of opposite systems, of the quadric, through the vertex. The cubic monoid, in this case, breaks up into the quadric and a plane through the vertex. The "quartic of the first kind" may have an actual double point or cusp, in which case the cone has an additional double or cuspidal edge that meets the quadric only once at this double point or cusp and once at the vertex, and therefore does not lie on it. The cone may have a cuspidal edge due to an apparent cusp on the quartic curve, i. e. if a tangent to the curve passes through the vertex, the curve when viewed from the vertex appears to have a cusp on this tangent, which is therefore a cuspidal edge of the cone (the apparent cusp replaces one of the apparent double points and the cuspidal edge is one of the generators of the quadric). If two tangents to the quartic curve pass through the vertex, the cone has two cuspidal edges and the curve has two apparent cusps. When the quadric that cuts out the quartic goes through the vertex the residual consists entirely of the two double or cuspidal edges on which the apparent double points or apparent cusps lie, but for every other quadric that passes through the "quartic of the first kind," the residual is another "quartic of the first kind" similar to the original quartic; this may be shown as follows: since the quadric does not go through the vertex no edge or multiple can lie on it, and therefore the residual cannot break up, i. e. it must be a twisted quartic; moreover, every edge or multiple

* This is the number of common edges of its superior and inferior cones. Cayley, Collected Papers, V. p. 8.

edge meets the quadric twice and no more; then every double edge on which an apparent double point of the original quartic lies is met by the original quartic in two distinct points, and therefore the residual quartic must cross this double edge at the same two points, since there are two and only two branches of the complete intersection at these two points where the double edge meets the quadric; therefore, for every apparent double point of the original quartic there is an apparent double point of the residual quartic. The double or cuspidal edge, on which an actual double point or cusp of the original quartic lies, meets the quadric only once there, and must meet it at one more point, which is, therefore, a double point or cusp on the residual quartic. A cuspidal edge due to an apparent cusp on the original quartic meets the quadric in two consecutive points, and the residual quartic must pass through these two consecutive points, i. e. it has this cuspidal edge as a tangent and therefore the residual quartic also has an apparent cusp at this point. The residual quartic is therefore a "quartic of the first kind" similar to the original quartic. Each of these two "quartics of the first kind" considered as lying on the quadric, meets the generators of each system of the quadric in two points, and is therefore a 4_2 on the quadric; the formula for the number of intersections of two curves a_α and b_β on a quadric being $(a_\alpha, b_\beta) = a\,\beta + b\,\alpha - 2\,\alpha\,\beta$, the two quartics in question, considered as lying on the quadric, intersect in $(4_2, 4_2) = 8 + 8 - 8 = 8$ points; but on the cone these quartics are 4_1's, and, since they do not go through the vertex, formula (1) gives the total number of their intersections, so that regarding these quartics as lying on the cone they intersect in $(4_1, 4_1) = 4 + 4 - 4 = 4$ points. This illustrates what was said (p. 25) in reference to the point of crossing of two curves on a multiple line not counting as a point of intersection of those curves, considered as lying on the surface having the multiple line, when those curves lie on different sheets at this point of crossing. In the present case we have four such points, two on each of the two double or cuspidal edges on which the apparent double points or cusps lie; these four points count as points of intersection of the quartics considered as lying on the quadric, because the quartics lie on the same sheet of the quadric, but they do not count as points of intersection of the quartics, considered as lying on the cone, because these curves lie on different sheets of the cone at these points; these four points make up the difference in the number of intersections that the quartics have on the two surfaces.

The "quartic of the first kind" may lie on a cone with a tac-nodal edge formed by the union of two double or cuspidal edges; the quar-

tic then has an apparent tac-node equivalent to two apparent double points.

The quartic that lies on the cone having a triple edge, is a "quartic of the second kind;" for, if it were a "quartic of the first kind," we could pass a quadric through it and through the vertex; the triple line would then lie on the quadric and be a generator of one system; the generator of the other system that passes through the vertex would meet the cone four times at the vertex and at least once on the curve, and would therefore coincide with an edge of the cone; but an edge of the cone meets the quartic once only, and therefore the quartic would be met by the generators of one system of the quadric once only, and would not be a "quartic of the first kind" as supposed.

Every "quartic of the second kind" has three apparent double points (or cusps), and cannot lie on a quartic cone with fewer than three double or cuspidal edges; two of these may unite, forming a tac-nodal edge, or all three double edges may unite, forming a triple edge; on the tac-nodal edge the quartic has an apparent tac-node equivalent to two apparent double points, and on the triple edge the quartic has an apparent triple point equivalent to three apparent double points. Through every "quartic of the second kind" can be passed one and only one quadric, and, if the quartic lies on a cone with a triple edge, the quadric always passes through the vertex; for, the triple line meets the quadric three times on the quartic curve, and therefore lies on it, being a generator of one system of the quadric; the generator of the other system that passes through the vertex coincides with an edge of the cone, as we have seen, and therefore the residual consists of this edge and the triple edge. When the "quartic of the second kind" lies on a cone with three double (or cuspidal) edges, the quadric cannot go through the vertex (for if it did all three double or cuspidal edges would lie on it), and the residual is therefore another "quartic of the second kind" because it has an apparent double point (cusp) on each of the three double (cuspidal) edges; the points of crossing are the same as those of the original quartic, and the curves lie on different sheets at these points. Now, the generators of the quadric meet the cone four times, and, considering one system of generators, they must meet the cone three times on one quartic and once on the other quartic (since every "quartic of the second kind" on a quadric meets the generators of one system three times and those of the other system once), i. e. on the quadric, one of the quartics is a 4_3, and the other is a 4_1; therefore, considered as lying on the quadric, these quartics intersect in 10 points, $(4_3, 4_1) = 4 + 12 - 6 = 10$. But on the cone each

quartic is a 4_1, and these two quartics intersect in only 4 points, $(4_1, 4_1) = 4$. The six points, two on each double or cuspidal edge, where the branches of the two curves cross, do not count as points of intersection of the two quartics considered as lying on the cone, for the branches lie on different sheets of the cone at these six points, and this accounts for the difference in the number of intersections that the curves have on the two surfaces, — a further illustration of the principle already stated.

The different species of twisted quartics that lie on the different cones can be tabulated as follows, where δ is the number of actual double points, κ the number of actual cusps, h the number of apparent double points, k the number of apparent cusps, and T the number of apparent triple points.

Quartic Cone.			Quartic Curves.						
			First Kind.				Second Kind.		
Double Edges.	Cuspidal Edges.	Triple Edges.	δ	κ	h	k	h	k	T
2	0	0	0	0	2	0			
1	1	0	0	0	1	1			
0	2	0	0	0	0	2			
2	1	0	0	1	2	0	2	1	0
			1	0	1	1			
1	2	0	1	0	0	2	1	2	0
			0	1	1	1			
3	0	0	1	0	2	0	3	0	0
0	3	0	0	1	0	2	0	3	0
0	0	1					0	0	1

5. Salmon* has divided twisted quintics into four groups, viz. group (I), having four apparent double points, group (II), having five appar-

* Geom. of Three Dimensions, § 352.

ent double points, and groups (III) and (IV), having six apparent double points. Now, from an ordinary point on any curve of order m, the number of apparent double points of the curve is $h - m + 2$,* where h is the number of apparent double points of the curve from an arbitrary point; therefore, since every twisted quintic on a quartic cone has one branch through the vertex, the number of apparent double points from the vertex, i. e. the number of double edges of the quartic cone, is $h - 3 \geqq 1$; moreover, since any twisted quintic can be cut out by a cubic monoid, the six lines of the monoid that pass through the vertex must count for seven, the order of the residual, and therefore the cone has at least one double (or cuspidal) edge with which one of these six lines coincides; therefore there is no quintic curve on the non-singular cone. There is a twisted quintic, which is not a special case of any of the groups given by Salmon, that has only three apparent double points, but it has an actual triple point, where the three tangents do not lie in the same plane, and the quartic cone on which it lies has a triple edge due to this actual triple point. All the quartic cones, except the non-singular cones, have species of twisted quintics on them, and these may be tabulated in the same way as the twisted quartics.

We have seen that any twisted sextic can be cut out by a cubic monoid, and, since the residual is of order six, the six lines of the monoid that pass through the vertex may be six edges of the cone, and therefore the non-singular cone, as well as each of the other cones, may have twisted sextics on it.

Any twisted curve of order 7 can be cut out by a quartic monoid which has 12 lines that pass through the vertex, 9 of which may be edges of the cone, forming the total residual, and therefore a quartic cone of any group has on it some species of twisted curve of order 7. For $a \geqq 8$ we have $a \geqq 2$ for some species of a_a, and therefore when $a \geqq 8$, a quartic cone of any group has on it some species of a_a.

* Salmon's Geom. of Three Dimensions, § 330, example 2.